

Equitable Coloring of Interval Graphs and Products of Graphs*

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Abstract

We confirm the equitable Δ -coloring conjecture for interval graphs and establish the monotonicity of equitable colorability for them. We further obtain results on equitable colorability about square (or Cartesian) and cross (or direct) products of graphs.

1 Introduction

All graphs $G = (V, E)$ considered in this paper are finite, loopless, and without multiple edges. Let C_n and K_n denote the cycle and the complete graph on n vertices, respectively. We also use $K_{x,y}$ (or $K_{x,y,z}$) to denote the complete bipartite (or tripartite) graph with parts of sizes x and y (or x , y , and z). A graph G is said to be k -colorable if there is a function $c : V(G) \rightarrow [k] = \{0, 1, \dots, k-1\}$ such that adjacent vertices are mapped to distinct numbers. The function c is called a proper k -coloring of G . All pre-images of a fixed number form a so-called *color class*. Each color class is an independent set, i.e., no two vertices in the same color class are adjacent. The smallest number k such that G is k -colorable is called the *chromatic number* of G , denoted by $\chi(G)$. A graph G is said to be *equitably k -colorable* if there is a proper k -coloring whose color classes V_0, V_1, \dots, V_{k-1}

*This paper was originally contributed to a hitherto unpublished Festschrift in honor of Man Keung Siu in February 2004.

satisfy the condition $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. The smallest integer n for which G is equitably n -colorable is called the *equitable chromatic number* of G , denoted by $\chi_=(G)$. This notion of equitable colorability was first introduced in Meyer [8]. It is evident that $\chi(G) \leq \chi_=(G)$.

Hajnal and Szemerédi [4] shows that a graph G is equitably k -colorable if $k \geq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G . So we may define the parameter $\chi_*(G)$ of G , called the *equitable chromatic threshold*, to be the smallest integer n such that G is equitably k -colorable for all $k \geq n$. Thus $\chi_*(G) \leq \Delta(G) + 1$. It is obvious that k -colorability is monotone in the sense that G is k -colorable once $k \geq \chi(G)$. However, equitable k -colorability may fail to be monotone. The complete bipartite graph $K_{2m+1, 2m+1}$ provides an example showing $\chi(K_{2m+1, 2m+1}) = \chi_=(K_{2m+1, 2m+1}) = 2 < \chi_*(K_{2m+1, 2m+1}) = 2m + 2$. The following conjecture proposed by Chen, Lih, and Wu [1] still remains open.

The equitable Δ -coloring conjecture. Let G be a connected graph. If G is not a complete graph, or an odd cycle, or a complete bipartite graph $K_{2m+1, 2m+1}$, then G is equitably $\Delta(G)$ -colorable.

We refer the reader to a survey on equitable colorability by Lih [6] for relevant concepts and results. The present paper supplies proofs of some statements announced in [6]. In section 2, we will confirm the equitable Δ -coloring conjecture for interval graphs and establish the monotonicity of equitable colorability for them. Sections 3 and 4 will handle equitable colorability of two types of graph products, namely, the square and the cross products.

2 Interval graphs

A graph $G(V, E)$ is called an *interval graph* if there exists a family $\{I_v \mid v \in V(G)\}$ of intervals on the real line such that u and v are adjacent vertices if and only if $I_u \cap I_v \neq \emptyset$. Such a family $\{I_v \mid v \in V(G)\}$ is commonly referred to as an *interval representation* of G . Instead of intervals of real numbers, these intervals may be replaced by finite intervals on a linearly ordered set.

A *clique* of a graph G is a complete subgraph Q of G such that no complete subgraph of G contains Q as a proper subgraph. For an interval graph G , Gillmore and Hoffman [3] shows that its cliques can be linearly ordered as $Q_0 < Q_1 < \dots < Q_m$ so that for every vertex v of G the cliques containing v occur consecutively. We assign the finite interval $I_v = [Q_i, Q_j]$ in this linear order to the vertex v if all the cliques containing v are precisely Q_i, Q_{i+1}, \dots, Q_j . Again u and v are adjacent if and only if $I_u \cap I_v \neq \emptyset$. We call this representation of G a *clique path* representation of G . Conversely, the existence of a clique path representation implies that the graph is an interval graph.

Once a clique path representation is given, we let $\text{left}(v)$ and $\text{right}(v)$ stand for the left and right endpoint, respectively, of the interval I_v . Then the following linear order

on the vertices of G can be defined. We let $u < v$ if $(\text{left}(u) < \text{left}(v))$ or $(\text{left}(u) = \text{left}(v)$ and $\text{right}(u) < \text{right}(v))$. If u and v have the same left and right endpoints, we choose $u < v$ arbitrarily. For any three vertices u , v , and w of G , this linear order satisfies the following condition.

$$\text{If } u < v < w \text{ and } uw \in E(G), \text{ then } uv \in E(G). \quad (1)$$

Olariu [9] shows that the existence of a linear order satisfying (1) characterizes interval graphs.

Theorem 1 *Let G be a connected interval graph on n vertices. If G is not a complete graph, then G is equitably $\Delta(G)$ -colorable.*

Proof. From a clique path representation of G , we linearly order the vertices of G into $v_0 < v_1 < \dots < v_{n-1}$ as defined above to satisfy condition (1). Let $(a \bmod b)$ denote the remainder of a divided by b . Define $c(v_i) = (i \bmod \Delta(G))$ for all $v_i \in V(G)$. It is evident that the range of c contains $\Delta(G)$ colors and the pre-images of any two colors have sizes differing by at most one. Suppose that $v_i < v_j$ and $c(i) = c(j)$ for a pair of adjacent vertices v_i and v_j . It follows that $j = i + k\Delta(G)$ for some positive integer k . Condition (1) implies that $k \not\geq 1$ and v_i is adjacent to $\Delta(G)$ vertices that are greater than v_i . However, the connectedness of G implies that v_i is adjacent to at least one smaller vertex unless $i = 0$. Since the degree of v_i is at most $\Delta(G)$, it follows that the neighbors of $v_i = v_0$ are precisely $v_1, v_2, \dots, v_{\Delta(G)}$.

We claim that G would be a complete graph on the vertices $v_0, v_1, \dots, v_{\Delta(G)}$. Since $v_0 < v_1$, either $\text{left}(v_0) = \text{left}(v_1)$ or there should be a vertex u in the clique $\text{left}(v_0)$ such that $v_0 < u < v_1$. However, the latter is impossible. We hence further have $\text{right}(v_0) \leq \text{right}(v_1)$. This implies that v_2 is adjacent to v_1 since v_2 is adjacent to v_0 . Reasoning as before, we can show that $\text{left}(v_1) = \text{left}(v_2)$ and $\text{right}(v_1) \leq \text{right}(v_2)$. Arguing inductively in this way, all the vertices $v_0, v_1, \dots, v_{\Delta(G)}$ are shown to be mutually adjacent. Since G is connected and each vertex in $v_0, v_1, \dots, v_{\Delta(G)}$ has degree $\Delta(G)$, our claim is true. However, this consequence is contradicted by our assumptions. We conclude that c is a proper coloring. \square

The above proof can be modified in a straightforward manner to establish the following.

Corollary 2 *Let G be a disconnected interval graph. If $\omega(G)$, the largest size of a clique of G , is at most $\Delta(G)$, then G is equitably $\Delta(G)$ -colorable.*

Theorem 3 *Let G be an interval graph. Then $\chi_=(G) = \chi_*(G)$.*

Proof. Let G have n vertices. Suppose that c is an equitable k -coloring of G . Let V_0, V_1, \dots, V_{k-1} be the color classes of c such that $|V_j| = \lceil \frac{n-j}{k} \rceil$ for all $j \in [k]$. We are going to modify c to get an equitable $(k+1)$ -coloring of G by the following algorithm.

Input. The vertices of G are listed from left to right satisfying condition (1).

Output. The new color classes V_0, V_1, \dots, V_k are produced so that $|V_j| = \lceil \frac{n-j}{k+1} \rceil$ for all $j \in [k+1]$.

Initialization. Let $S \leftarrow \{m \mid 0 \leq m \leq k-1 \text{ and } |V_m| > \lceil (n-m)/(k+1) \rceil\}$, $G_0 \leftarrow G \setminus \bigcup \{V_m \mid 0 \leq m \leq k-1 \text{ and } m \notin S\}$, $V_k \leftarrow \emptyset$, and $i \leftarrow 0$. (The sequence S records which old color classes have not been reduced to the proper size.)

Procedure.

1. If $S = \emptyset$, then STOP; else do the following.
2. Examine each vertex of G_i from left to right. While the color of a vertex occurs the first time in G_i , mark that vertex. Let v be the first vertex such that $c(v) = c(u)$ for a unique $u < v$.
3. Let $V_k \leftarrow V_k \cup \{u\}$ and $V_{c(v)} \leftarrow V_{c(v)} \setminus \{u\}$.
4. If $|V_{c(v)}| = \lceil \frac{n-c(v)}{k+1} \rceil$, then $G_{i+1} \leftarrow G_i \setminus (\{\text{all marked vertices}\} \cup V_{c(v)})$ and $S \leftarrow S \setminus \{c(v)\}$; else $G_{i+1} \leftarrow G_i \setminus \{\text{all marked vertices}\}$.
5. Let $i \leftarrow i + 1$ and GOTO 1.

Now we want to prove that this algorithm is correct.

We claim that all the vertices brought to V_k are non-adjacent. Suppose on the contrary that there are adjacent vertices x and y in V_k . We may let x be brought to V_k earlier than y . From our procedure, it implies that $x < y$ in the linear order of G . When x was brought into V_k , there was a vertex z such that $x < z$ and they both were in the same color class. The vertices appearing earlier than z were all excluded from further consideration by our procedure. So we must have $z < y$. If x and y are neighbors, then condition (1) implies that x and z are neighbors, which is impossible.

Since the index j is deleted from S just as $|V_j| = \lceil \frac{n-j}{k+1} \rceil$ and since $n = \sum_{j=0}^k \lceil \frac{n-j}{k+1} \rceil$, our procedure stops if and only if we have obtained $|V_j| = \lceil \frac{n-j}{k+1} \rceil$ for all $j \in [k+1]$.

When we start examining G_i , each old color class possesses at most i marked vertices. This is true because no two marked vertices have the same color in each round. Suppose that S is nonempty when we start examining G_i . Then $|V_j| > \lceil \frac{n-j}{k+1} \rceil$ for all $j \in S$. Since after each looping of our procedure the size of V_k is increased by one, we know that $|V_k| = i < \lceil \frac{n-k}{k+1} \rceil \leq \lceil \frac{n-j}{k+1} \rceil < |V_j|$ by our termination criterion above. Therefore $V_j \cap G_i$ contains at least two unmarked vertices for every $j \in S$ and the execution of step 2 of our procedure can continue. \square

For a special subclass of interval graphs, the above monotonicity of equitable coloring starts right from the chromatic number. If an interval representation of an interval graph G can be found so that each interval is of unit length, then G is called a *unit interval* graph. A unit interval graph can be equivalently characterized as a *claw-free* interval graph, i.e., an interval graph containing no $K_{1,3}$ as an induced subgraph. A result of de Werra [11] implicitly implies that every claw-free graph is equitably k -colorable for all $k \geq \chi(G)$.

We now supply a simple algorithm for constructing an equitable $\chi(G)$ -coloring for a unit interval graph G .

The vertices of a unit interval graph G can be linearly ordered $v_0 < v_1 < \dots < v_n$ such that each clique of G consists of consecutive vertices ([7]). Define $c(v_i) = (i \bmod \omega(G))$ for all $v_i \in V(G)$. It is evident that the range of c contains $\omega(G)$ colors and the pre-images of any two colors have sizes differing by at most one. Suppose that $v_i < v_j$ and $c(v_i) = c(v_j)$ for a pair of adjacent vertices v_i and v_j . It follows that $j = i + k\omega(G)$ for some positive integer k . This would imply that the set $\{v_i, v_{i+1}, \dots, v_j\}$, whose size is at least $\omega(G) + 1$, is included in a clique. It follows from this contradiction that c is a proper coloring of G and $\chi_=(G) \leq \omega(G)$. Since interval graphs are perfect graphs, we have $\omega(G) = \chi(G)$.

3 Square products

The *square product*, also known as the *Cartesian product*, of graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ has vertex set $\{(u, v) \mid u \in V_1 \text{ and } v \in V_2\}$ such that $\{(u, x), (v, y)\}$ is an edge if and only if $(u = v \text{ and } xy \in E_2)$ or $(x = y \text{ and } uv \in E_1)$. We denote the square product by $G_1 \square G_2$.

Theorem 4 *If both G_1 and G_2 are equitably k -colorable, then $G_1 \square G_2$ is also equitably k -colorable.*

Proof. Let U_0, U_1, \dots, U_{k-1} and V_0, V_1, \dots, V_{k-1} denote the color classes of G_1 and G_2 , respectively. Suppose that we have $|U_0| = |U_1| = \dots = |U_a| = \alpha$ for some $a \in [k]$ and the other color classes of G_1 are of size $\alpha - 1$. Similarly, suppose that $|V_0| = |V_1| = \dots = |V_b| = \beta$ for some $b \in [k]$ and the other color classes of G_2 are of size $\beta - 1$.

In the first stage, we are going to construct an auxiliary Latin square $L = (a_{ij})$ of order k , using the numbers in $[k]$ as entries. Let $q = \gcd(b, k)$ and $p = k/q$. So we may write $b = mq$ for some m such that $\gcd(m, p) = 1$. We use elements of $[k]$ to index the rows and columns of L . The (i, j) -entry of L is defined to be $a_{i,j} = (ib + \lfloor i/p \rfloor + j \bmod k)$.

Suppose that $a_{i,j} = ib + j = i'b + j' = a_{i',j'}$ for $0 \leq i, i' \leq p - 1$ and $0 \leq j, j' \leq q - 1$. Since q divides both b and k , it follows that q divides $j - j'$, and hence $j = j'$. This in turn implies that p divides $(i - i')m$, which is impossible. So the upper left $p \times q$ corner of L , denoted by L' , is filled up with the numbers in $[k]$, each occurring exactly once. We observe that $a_{i,0} = ib + \lfloor i/p \rfloor = rb + s = a_{r,s}$ if $i = sp + r$, where $0 \leq r \leq p - 1$ and $0 \leq s \leq q - 1$. This means that the first column of L is a concatenation of the successive columns of L' , hence contains no repeated numbers. As each row of L is a cyclic exhibition of the numbers in $[k]$, no repetitions in the first column imply that L is a Latin square.

Now we divide L into four subsquares $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so that the upper left corner A is of order $a \times b$. We observe that, if $sp \leq i \leq sp + p - 2$, then (i) $a_{sp+p-1, b-1} + 1 = a_{sp, 0}$; (ii) $a_{i, b-1} + 1 = a_{i+1, 0}$. The sequence obtained by concatenating the $sp, sp + 1, \dots, sp + p - 1$

rows of A , $0 \leq s \leq q-1$, is a cyclic exhibition of the numbers in $[k]$, starting with the entry at $a_{sp,0}$ and each number occurring exactly m times. It implies that each number in $[k]$ occurs in A precisely t or $t-1$ times, where $t = \lceil ab/k \rceil$.

Next we let $W_z = \bigcup \{U_i \square V_j \mid a_{i,j} = z\}$ for each $z \in [k]$. It follows from the definition of a square product that the following properties hold in $G_1 \square G_2$.

- (1) Every $U_i \square V_j$ is an independent set.
- (2) Every $(U_{i_1} \square V_{j_1}) \cup (U_{i_2} \square V_{j_2})$ is an independent set if $i_1 \neq i_2$ and $j_1 \neq j_2$.

Consequently, every W_z is an independent set in $G_1 \square G_2$. The numbers of occurrence of z in the subsquares A , B , C , and D belong to two types: $(t, a-t, b-t, k-a-b+t)$ or $(t-1, a-t+1, b-t+1, k-a-b+t-1)$. If W_z belongs to the first type, then $|W_z| = t\alpha\beta + (a-t)(\alpha\beta - \alpha) + (b-t)(\alpha\beta - \beta) + (k-a-b+t)(\alpha\beta - \alpha - \beta + 1)$. If W_z belongs to the second type, then $|W_z| = (t-1)\alpha\beta + (a-t+1)(\alpha\beta - \alpha) + (b-t+1)(\alpha\beta - \beta) + (k-a-b+t-1)(\alpha\beta - \alpha - \beta + 1)$. The difference between the two sizes is precisely one. We conclude that W_0, W_1, \dots, W_{k-1} form equitable color classes for $G_1 \square G_2$. \square

Corollary 5 *Let G_1 have n vertices and G_2 be n -colorable. Then $G_1 \square G_2$ is equitably n -colorable.*

Proof. Let the vertex set of G_1 be $\{u_0, u_1, \dots, u_{n-1}\}$. Since G_2 is n -colorable, let V_0, V_1, \dots, V_{n-1} be a set of color classes. Define $U_k = \bigcup \{\{u_i\} \times V_j \mid j - i \equiv k \pmod{n}\}$ for $0 \leq k \leq n-1$. Thus each U_k is an independent set in $G_1 \square G_2$ and $|U_k|$ is equal to the order of G_2 . \square

Corollary 6 *Let $G = G_1 \square G_2 \square \dots \square G_n$, where each G_i is a path, a cycle, or a complete graph. Then we have $\chi(G) = \chi_=(G) = \chi_*(G) = \max\{\chi(G_i) \mid 1 \leq i \leq n\}$.*

Proof. This statement follows from Theorem 4 together with the well-known fact ([10]) that $\chi(G) = \max\{\chi(G_i) \mid 1 \leq i \leq n\}$. \square

Corollary 7 *We have $\chi_*(G_1 \square G_2) \leq \max\{\Delta(G_1) + 1, \Delta(G_2) + 1\}$.*

Proof. Since $\chi_*(G_1) \leq \Delta(G_1) + 1$ and $\chi_*(G_2) \leq \Delta(G_2) + 1$, we have $\chi_*(G_1 \square G_2) \leq \max\{\Delta(G_1) + 1, \Delta(G_2) + 1\}$ by Theorem 4. \square

Corollary 8 *Suppose that G_1 and G_2 are graphs each with at least one edge. Then $G_1 \square G_2$ is equitably $\Delta(G_1 \square G_2)$ -colorable.*

Proof. Since neither G_1 nor G_2 consists of isolated vertices, we have $\Delta(G_1 \square G_2) \geq \max\{\Delta(G_1) + 1, \Delta(G_2) + 1\}$. Corollary 7 implies that $\chi_*(G_1 \square G_2) \leq \max\{\Delta(G_1) + 1, \Delta(G_2) + 1\} \leq \Delta(G_1 \square G_2)$. \square

If we weaken the assumption on G_1 in Theorem 4 to that of its k -colorability, then the conclusion may not follow. Let $K_{1,5}$ denote the star graph on 6 vertices and P_3 the

path on 3 vertices. The cross product $K_{1,5} \square P_3$ is a bipartite graph with one part A of size 7 and the other part B of size 11. Let us consider any proper 2-coloring of this product. Since there is a vertex x in B that is adjacent to every vertex in A , none of the vertices in A belong to the color class containing x . But any vertex in B is adjacent to some vertex in A . Therefore, this 2-coloring cannot be equitable. This example shows that, even if $\chi(G_1) = \chi_=(G_2) = k$, the product $G_1 \square G_2$ may not be equitable k -colorable.

If we assume that $\chi_=(G_1) = \chi_=(G_2) = k$, it may not lead to the conclusion $\chi_=(G_1 \square G_2) = k$. Let us consider $K_{1,2n} \square K_{1,2n}$. Let the vertex set of $K_{1,2n}$ be $\{a_0, a_1, \dots, a_{2n}\}$ so that a_0 is the vertex of degree $2n$. It is easy to see that $\chi_=(K_{1,2n}) = n + 1$. The following array gives an equitable 4-coloring of $K_{1,2n} \square K_{1,2n}$. (The entry at position (i, j) is the color given to the vertex (a_i, a_j) .)

$$\begin{array}{c}
 \begin{array}{c} n \\ \left\{ \begin{array}{c} 0 \\ 3 \\ \cdot \\ 3 \\ 1 \\ \cdot \\ 1 \end{array} \right. \end{array}
 \begin{array}{c} \overbrace{\begin{array}{ccc} 1 & \cdots & 1 \end{array}}^n \\ 0 \quad \cdots \quad 0 \\ \cdot \quad \cdots \quad \cdot \\ 0 \quad \cdots \quad 0 \\ 2 \quad \cdots \quad 2 \\ \cdot \quad \cdots \quad \cdot \\ 2 \quad \cdots \quad 2 \end{array}
 \begin{array}{c} 2 \\ 0 \\ \cdot \\ 0 \\ 3 \\ \cdot \\ 3 \end{array}
 \begin{array}{c} \overbrace{\begin{array}{ccc} 2 & \cdots & 2 \end{array}}^{n-1} \\ 1 \quad \cdots \quad 1 \\ \cdot \quad \cdots \quad \cdot \\ 1 \quad \cdots \quad 1 \\ 3 \quad \cdots \quad 3 \\ \cdot \quad \cdots \quad \cdot \\ 3 \quad \cdots \quad 3 \end{array}
 \end{array}$$

The following example shows that $\chi_=(G_1 \square G_2) \leq \max\{\chi_=(G_1), \chi_=(G_2)\}$ is false in general. Let $G_1 = K_{3,3}$ and $G_2 = K_{2,1,1}$. We have $\chi_=(G_1) = 2$ and $\chi_=(G_2) = 3$, but $\chi_=(G_1 \square G_2) = 4$. It is easy to see that $G_1 \square G_2$ is equitably 4-colorable. We want to show that it is not equitably 3-colorable. We write the vertices of G_1 into a sequence $[u_0, u_1, u_2, v_0, v_1, v_2]$ so that $\{u_0, u_1, u_2\}$ and $\{v_0, v_1, v_2\}$ form independent sets, respectively. We write the vertices of G_2 into a sequence $[a_0, a_1, b, c]$ so that, except a_0 and a_1 , all pairs of vertices are adjacent. Now we arrange the vertices of $G_1 \square G_2$ into a 6×4 array. Suppose that there were an equitable 3-coloring of this array. Thus every color class contains exactly 8 vertices. Each pair (x, a_0) and (x, a_1) must have the same color since they are adjacent to the two endpoints of the edge $(x, b)(x, c)$. It implies that the first column has at least two colors. Since we cannot have a pair (u_i, a_0) and (v_j, a_0) with the same color, either all (u_i, a_0) 's are of the same color or all (v_i, a_0) 's are of the same color. Either possibility implies that some color class would contain 9 vertices.

In general, let G_1 be equitably k_1 -colorable and G_2 be equitably k_2 -colorable. It remains open to find conditions that force $G_1 \square G_2$ to be equitably $(\max\{k_1, k_2\})$ -colorable.

4 Cross products

The *cross product*, also known as the *direct product*, of graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ has vertex set $\{(u, v) \mid u \in V_1 \text{ and } v \in V_2\}$ such that $\{(u, x), (v, y)\}$ is an edge if and only if $uv \in E_1$ and $xy \in E_2$. We denote the cross product by $G_1 \times G_2$.

Lemma 9 We have $\chi_=(G_1 \times G_2) \leq \min\{|V(G_1)|, |V(G_2)|\}$.

Proof. Let $V(G_1) = \{u_0, u_1, \dots, u_m\}$ and $U_i = \{u_i\} \times V(G_2)$ for all $0 \leq i \leq m$. Then U_i is an independent set of $G_1 \times G_2$ and $|U_i| = |V(G_2)|$ for every $0 \leq i \leq m$. Thus $\chi_=(G_1 \times G_2) \leq |V(G_1)|$. Similarly, we have $\chi_=(G_1 \times G_2) \leq |V(G_2)|$. \square

Corollary 10 We have $\chi_=(K_m \times K_n) = \min\{m, n\}$.

Proof. Duffus, Sands, and Woodrow [2] shows that $\chi(K_m \times K_n) = \min\{\chi(K_m), \chi(K_n)\}$. Then Lemma 9 implies the result. \square

We note that $\chi_*(G_1 \times G_2) \leq \min\{|V(G_1)|, |V(G_2)|\}$ is false in general. For instance, $K_2 \times K_n$ is not equitably $(\frac{n+1}{2})$ -colorable if $n > 1$ and $n \equiv 1 \pmod{4}$.

Let $n = 4k + 1$ for some $k \geq 1$. We observe that $K_2 \times K_n$ is equal to $K_{n,n} - M$ for a complete matching M . If there were an equitable $(\frac{n+1}{2})$ -coloring of $K_2 \times K_n$, then there would be two color classes of size 3 and $2k - 1$ color classes of size 4. Any color class of size 3 cannot contain vertices in both parts. Yet neither $4k - 2$ nor $4k - 5$ is divisible by 4. Hence the desired equitable color classes cannot exist.

We would surmise that $\chi_*(G_1 \times G_2) \leq \max\{|V(G_1)|, |V(G_2)|\}$ should be true.

Theorem 11 Let $m, n \geq 3$. Then

$$\chi_=(C_m \times C_n) = \chi_*(C_m \times C_n) = \begin{cases} 2, & \text{if } mn \text{ is even;} \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let C_m be the cycle $u_0 u_1 \dots u_{m-1} u_0$ and C_n be the cycle $v_0 v_1 \dots v_{n-1} v_0$. We note that $C_m \times C_n$ is a 4-regular graph. Hence it is equitably k -colorable for all $k \geq 5$.

Case 1. We use two colors.

If mn is even, then $C_m \times C_n$ is a bipartite graph with parts of equal size. Hence $\chi_=(C_m \times C_n) = 2$. If $m \leq n$ are both odd, then there exists an odd cycle in $C_m \times C_n$: $(u_0, v_0)(u_1, v_1) \dots (u_{m-1}, v_{m-1})(u_{m-2}, v_m)(u_{m-1}, v_{m+1}) \dots (u_{m-1}, v_{n-1})(u_0, v_0)$. Hence $\chi_=(C_m \times C_n) \geq 3$.

Case 2. We use three colors.

It is straightforward to verify the colorings to be defined in the following subcases are equitable 3-colorings of $C_m \times C_n$.

Subcase 2.1. Assume that m or n , say n , is divisible by 3. Define the coloring $a(u_i, v_j) = (j \bmod 3)$.

Subcase 2.2. Assume that $m - 1$ or $n - 1$, say $n - 1$, is divisible by 3. Also assume that $n > 4$. Define the coloring

$$b(u_i, v_j) = \begin{cases} 0, & \text{if } j = n - 2; \\ 1, & \text{if } j = n - 1 \text{ or } (j = n - 4 \text{ and } i < \lceil m/3 \rceil); \\ 2, & \text{if } j = n - 3 \text{ or } (j = 0 \text{ and } i < \lfloor m/3 \rfloor); \\ a(u_i, v_j), & \text{otherwise.} \end{cases}$$

Subcase 2.3. Assume that $m - 2$ or $n - 2$, say $n - 2$, is divisible by 3. Also assume that $n > 5$. Define the coloring

$$c(u_i, v_j) = \begin{cases} 0, & \text{if } j = n - 3; \\ 2, & \text{if } j = n - 2, \text{ or } (j = 0 \text{ and } i < \lceil m/3 \rceil), \\ & \text{or } (j = n - 4 \text{ and } i < \lfloor m/3 \rfloor); \\ a(u_i, v_j), & \text{otherwise.} \end{cases}$$

Subcase 2.4. There are three remaining cases that are solved by the following arrays of colorings.

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{array} \quad \begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 & 2 \end{array} \quad \begin{array}{ccccc} 2 & 0 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 2 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 \end{array}$$

Case 3. We use four colors.

Again it is straightforward to verify the colorings to be defined in the following subcases are equitable 4-colorings of $C_m \times C_n$.

Subcase 3.1. Assume that m or n , say n , is divisible by 4. Define $d(u_i, v_j) = (j \bmod 4)$.

Subcase 3.2. Assume that $m - 1$ or $n - 1$, say $n - 1$, is divisible by 4. Also assume that $n > 5$. Define the coloring

$$e(u_i, v_j) = \begin{cases} 0, & \text{if } j = n - 2; \\ 1, & \text{if } (j = n - 1 \text{ and } i \geq \lfloor (m + 3)/4 \rfloor) \\ & \text{or } (j = n - 5 \text{ and } i < \lceil m/2 \rceil); \\ 2, & \text{if } j = n - 5 \text{ and } i \geq \lceil m/2 \rceil + \lfloor m/4 \rfloor; \\ 3, & \text{if } j = n - 4 \text{ or } (j = n - 1 \text{ and } i < \lfloor (m + 3)/4 \rfloor); \\ d(u_i, v_j), & \text{otherwise.} \end{cases}$$

Subcase 3.3. Assume that $m - 2$ or $n - 2$, say $n - 2$, is divisible by 4. Define the coloring

$$f(u_i, v_j) = \begin{cases} 1, & \text{if } j = n - 2 \text{ and } i \geq \lfloor m/2 \rfloor; \\ 2, & \text{if } j = n - 1 \text{ and } i < \lfloor m/2 \rfloor; \\ 3, & \text{if } j = n - 1 \text{ and } i \geq \lfloor m/2 \rfloor; \\ d(u_i, v_j), & \text{otherwise.} \end{cases}$$

Subcase 3.4. Assume that $m - 3$ or $n - 3$, say $n - 3$, is divisible by 4. Define the coloring

$$g(u_i, v_j) = \begin{cases} 1, & \text{if } (j = n - 1 \text{ and } i \geq m - \lfloor (m + 2)/4 \rfloor); \\ & \text{or } (j = n - 3 \text{ and } i \geq m - \lfloor (m + 2)/4 \rfloor); \\ 3, & \text{if } j = n - 2 \text{ and } (i < \lceil m/4 \rceil \text{ or } i \geq \lceil m/2 \rceil); \\ d(u_i, v_j), & \text{otherwise.} \end{cases}$$

Subcase 3.5. There are three remaining cases that will be solved by the following arrays of colorings.

									0	1	2	0	1
0	1	1		0	0	2	1	3	0	1	2	3	1
0	3	3		0	1	2	1	2	0	1	2	3	1
0	2	2		0	3	3	1	2	0	3	2	3	2
									0	3	2	3	2

Theorem 12 *We have $\chi_=(K_n \times K_{n,n-1}) = \chi_=(K_n \times K_{n,n-1}) = n$.*

Proof. The statement is trivial when $n = 2$. Let us assume $n \geq 3$. Denote the vertices of K_n by u_0, u_1, \dots, u_{n-1} and the vertices of $K_{n,n-1}$ by disjoint parts: $A : a_0, a_1, \dots, a_{n-1}$ and $B : b_0, b_1, \dots, b_{n-2}$. We arrange vertices of $K_n \times K_{n,n-1}$ into an n by $2n - 1$ array so that the i -th row is equal to $(u_i, a_0)(u_i, a_1) \cdots (u_i, a_{n-1})(u_i, b_0)(u_i, b_1) \cdots (u_i, b_{n-2})$.

Claim 1. *The graph $K_n \times K_{n,n-1}$ is equitably k -colorable for all $k \geq n$.*

Let $k \geq n$. We are trying to equitably color $K_n \times K_{n,n-1}$ with k colors. The size of each color class should be m or $m + 1$, where $m = \lfloor n(2n - 1)/k \rfloor \leq 2n - 1$. If $m = 2n - 1$, then the n rows form an equitable n -coloring. Let us assume $m \leq 2n - 2$ and $\alpha(m + 1) + \beta m = n(2n - 1)$ for some α and β with $\alpha + \beta = k$. We are going to partition the vertices into independent sets of appropriate sizes and numbers.

We remove initial segments of length $m + 1$ from successive rows in a cyclic fashion. Once the α independent sets of size $m + 1$ have been removed, we partition the remaining part of each row into segments of length m . After all this is done, the number of vertices left in each row is less than m , hence the second coordinates all belong to B . All these leftover vertices form an independent set. We just partition them further into subsets of size m .

Claim 2. *The graph $K_n \times K_{n,n-1}$ is not equitably k -colorable for any $k < n$.*

Suppose that it were equitably k -colorable for some $k < n$. Then the size of each color class is at least $\lfloor n(2n - 1)/k \rfloor$. Now $n(2n - 1)/k \geq n(2n - 1)/(n - 1) = 2n + 1 + \frac{1}{n-1}$. It follows that $\lfloor n(2n - 1)/k \rfloor \geq 2n + 1$. If a color class contains two vertices whose second coordinates belong to different parts of $K_{n,n-1}$, then their first coordinates must equal. However, there are at most $2n - 1$ vertices with the same first coordinates. Hence the second coordinates of a color class must come from the same part of $K_{n,n-1}$.

Suppose that the part having n vertices is partitioned into x color classes and the part having $n - 1$ vertices is partitioned into y color classes. The sizes of color classes satisfy $|\lfloor n^2/x \rfloor - \lceil n(n - 1)/y \rceil| \leq 1$, which in turn implies $|n^2/x - n(n - 1)/y| \leq 1$. If $x \leq y$, then $n^2/x > (n^2 - n)/x + 1 \geq n(n - 1)/y + 1$. If $x > y$, then $n > 2y$. It follows that $(n + y - 1)(n - y) > n(y + 1)$, and hence $n(n - 1)/y > n^2/(y + 1) + 1 \geq n^2/x + 1$. \square

We note that, even if both G_1 and G_2 are equitably k -colorable, $G_1 \times G_2$ may not be equitably k -colorable. Let us consider $K_{m,m-1} \times K_{n,n-1}$. This is a disjoint union of

$K_{mn,(m-1)(n-1)}$ and $K_{(m-1)n,m(n-1)}$. If we properly color this union by two colors, then every part should be entirely colored with one color and two parts of the same connected component should be colored with different colors. However, the combined size of any two independent parts is different from $2mn - m - n$. Therefore, this disjoint union is not equitably 2-colorable. In particular, $\chi_{=}^*(K_{3,2} \times K_{3,2}) > 2$. (Actually, $\chi_{=}^*(K_{3,2} \times K_{3,2}) = 3$.) However, $\chi_{=}^*(K_{3,2}) = 2$ shows that the inequality $\chi_{=}^*(G_1 \times G_2) \leq \max\{\chi_{=}^*(G_1), \chi_{=}^*(G_2)\}$ is false in general.

We conclude this paper by posing the determination of the exact values for $\chi_{=}^*(K_m \times K_n)$ and $\chi_{=}^*(K_{m,m-1} \times K_{n,n-1})$ as an open problem.

References

- [1] B.-L. Chen, K.-W. Lih, P.-L. Wu, Equitable coloring and the maximum degree, *Europ. J. Combin.* 15 (1994) 443-447.
- [2] D. Duffus, B. Sands, R. E. Woodrow, On the chromatic number of the product of graphs, *J. Graph Theory* 9 (1985) 487-495.
- [3] P. C. Gilmore, A. J. Hoffman, Characterization of comparability graphs and interval graphs, *Canadian J. Math.* 16 (1964) 539-548.
- [4] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős, in: A. Rényi, V. T. Sós, (Eds.), *Combinatorial Theory and Its Applications*, Vol. 2, *Colloq. Math. Soc. János Bolyai* 4, North-Holland, Amsterdam, 1970, pp. 601-623.
- [5] W. Imrich, S. Klavžar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.
- [6] K.-W. Lih, The equitable coloring of graphs, in: D.-Z. Du, P. Pardalos (Eds.), *Handbook of Combinatorial Optimization*, Vol. 3, Kluwer, Dordrecht, 1998, pp. 543-566.
- [7] H. Maehara, On time graphs, *Discrete Math.* 32 (1980) 281-289.
- [8] W. Meyer, Equitable coloring, *Amer. Math. Monthly* 80 (1973) 920-922.
- [9] S. Olariu, An optimal greedy heuristic to color interval graphs, *Inform. Process. Lett.* 37 (1991) 21-25.
- [10] G. Sabidussi, Graphs with given group and given graph-theoretical properties, *Canad. J. Math.* 9 (1957) 515-525.
- [11] D. de Werra, Some uses of hypergraph in timetabling, *Asia-Pacific J. Oper. Res.* 2 (1985) 2-12.